

Some inequalities on hemi-slant product submanifolds in a cosymplectic manifold

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Abstract. Recently, M. Atcken studied Contact CR-warped product submanifolds in cosymplectic space forms and established general sharp inequalities for CR-warped products in a cosymplectic manifold [1]. In the present paper, we obtain an inequality for the squared norm of the second fundamental form in terms of constant ϕ -sectional curvature for hemi-slant products in cosymplectic manifolds. An inequality for hemi-slant warped products in a cosymplectic manifold is also given. The equality case is considered.

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1 Introduction

Bishop and O Neill [4] introduced the concept of warped products in 1969. They defined as follows

Definition 1.1. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and f a positive differentiable function on B . The warped product $B \times_f F$ of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F.$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q) , then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p) \|d\pi_2 U\|^2$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $B \times F$ on B and F , respectively.

They have given this important result for warped products

$$\nabla_U V = \nabla_V U = (U \ln f) V \quad (1.1)$$

for any vector fields U tangent to B and V tangent to F .

If the manifolds M_θ and M_\perp are slant and anti-invariant submanifolds respectively of a cosymplectic manifold \bar{M} , then their warped products are

- (a) $M_\perp \times_f M_\theta$,
- (b) $M_\theta \times_f M_\perp$.

In the sequel, we call the warped product submanifold (a) as warped product hemi-slant submanifold and the warped product (b) as hemi-slant warped product submanifold.

Recently, K. A. Khan et. al. [5] studied warped product semi-slant submanifolds in cosymplectic manifolds and proved that there does not exist warped product submanifold of the type $M_1 \times_f M_2$ of cosymplectic manifolds \bar{M} where M_1 and M_2 are any Riemannian submanifolds of \bar{M} with ξ tangential to M_2 other than Riemannian product. In [9] Siraj Uddin et. al. studied warped product submanifolds with slant factor and showed that warped product submanifold of the type $M_1 \times_f M_2$ of cosymplectic manifolds \bar{M} , such that $\xi \in TM_1$, where M_1 is totally real submanifold and M_2 is proper slant submanifold of \bar{M} are simply Riemannian product. So for this case we have established an inequality for the squared norm of the second fundamental form with constant ϕ -sectional curvature for cosymplectic manifolds.

On the other hand, warped product submanifold of the type $M_\theta \times_f M_\perp$ of cosymplectic manifolds \bar{M} , such that $\xi \in TM_\theta$, where M_θ is proper slant submanifold and M_\perp is totally real submanifold of \bar{M} , we have established an inequality for such type of submanifolds.

2 Preliminaries

Let \bar{M} be an almost contact metric manifold and let ϕ, ξ, η, g be it's almost contact metric structure. Thus \bar{M} is $(2n + 1)$ -dimensional differential manifold and ϕ, ξ, η, g are respectively, a $(1, 1)$ -tensor field, a vector field, a 1-form, a Riemannian metric on \bar{M} such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi) = 0, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi) \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y) \quad (2.2)$$

Here and in the sequel, X, Y, Z, \dots always denote arbitrary vector fields on \bar{M} if it is not otherwise stated. The fundamental 2-form Φ of \bar{M} is defined by $\Phi(X, Y) = g(\phi X, Y)$.

\bar{M} is said to be almost cosymplectic if the forms η and Φ are closed, that is, $d\eta = 0$ and $d\Phi = 0$, d being the operator of the exterior differentiation of differential forms [7]. If \bar{M} is almost cosymplectic and its almost contact structure (ϕ, ξ, η) is normal, then \bar{M} is called cosymplectic. It is well known that a necessary and sufficient condition for \bar{M} to be cosymplectic is that $\bar{\nabla}\phi$ vanishes identically, where $\bar{\nabla}$ is the Levi-Civita connection on \bar{M} . A plane section $\sigma \subset T_x(\bar{M})$ is a ϕ -section if σ is spanned by $\{u, \phi u\}$, for some $u \in T_x(\bar{M})$. If we restrict the ϕ -planes by a point function then the Riemannian sectional curvature ($of(\bar{M}, g)$) is the ϕ -sectional curvature. Now, let $\bar{M}(c)$ be a cosymplectic manifold of constant ϕ -sectional curvature c . Then the curvature tensor \bar{R} of $\bar{M}(c)$ is given by

$$(2.3) \quad \begin{aligned} \bar{R}(X, Y, Z, W) = & \frac{c}{4} \{ g(\phi Y, \phi Z)g(X, W) - g(\phi X, \phi Z)g(Y, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) \\ & + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) + 2g(X, \phi Y)g(\phi Z, W) \} \end{aligned}$$

for any $X, Y, Z, W \in \bar{M}(c)$.

Let M be a real m -dimensional submanifold of \bar{M} . We shall need the Gauss-Weigarten formulae

$$(2.4) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

for any $X, Y \in TM$ and $V \in T^\perp M$, where ∇^\perp is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_V is the Weingarten map associated with the vector field $V \in T^\perp M$ as

$$(2.5) \quad g(A_V X, Y) = g(h(X, Y), V).$$

We denote by \bar{R} and R the curvature tensor fields associated with $\bar{\nabla}$ and ∇ , respectively. We recall the equation of Gauss and Codazzi

$$(2.6) \quad \bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),$$

$$(2.7) \quad \bar{R}(X, Y, Z, V) = g((\nabla_X^h h)(Y, Z), V) - g((\nabla_Y^h h)(X, Z), V),$$

for any $X, Y, Z, W \in TM$ and $V \in T^\perp M$, where $(\nabla^h)h$ is the covariant derivative of the second fundamental form given by

$$(2.8) \quad (\nabla_X^h h)(Y, Z) = \bar{\nabla}_X h(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z),$$

for all $X, Y, Z \in TM$. The second fundamental form h satisfies the classical Codazzi equation (according to [6]) if

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z).$$

Let $p \in M$ and $\{e_1, \dots, e_m, \dots, e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p \bar{M}(c)$, such that e_1, \dots, e_m are tangent to M at p . We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{m} \sum_{i=1}^m h(e_i, e_i).$$

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, m\}, r \in \{m+1, \dots, 2m+1\}.$$

and

$$\|h\|^2 = \sum_{i,j=1}^m g((h(e_i, e_j), h(e_i, e_j))).$$

A submanifold M is totally geodesic in \bar{M} if $h = 0$, and minimal if $H = 0$.

3 Hemi-slant submanifolds

Throughout the section M is a hemi-slant submanifold of an almost contact metric manifold \bar{M} . Now in this section we shall discuss hemi-slant submanifolds of cosymplectic manifolds. More precisely, we will study integrability of the distributions of M and of the immersion of their leaves in M or \bar{M} .

Definition 3.1. [8] *A submanifold M is said to be a hemi-slant submanifold of an almost contact metric manifold \bar{M} , if there exist two orthogonal distributions D^\perp and D_θ on M such that*

- (i) $T_M = D^\perp \oplus D_\theta \oplus \langle \xi \rangle$.
- (ii) The distribution D^\perp is anti-invariant *i.e.*, $\phi D^\perp \subseteq T^\perp M$.
- (iii) The distribution D_θ is slant with slant angle $\theta \neq \pi/2$

from the definition it is clear that if $\theta = 0$, the hemi-slant submanifold become semi-invariant submanifold.

Suppose M to be a hemi-slant submanifold of an almost contact metric manifold \bar{M} . Then, for any $X \in TM$, put

$$(3.1) \quad X = P_1 X + P_2 X + \eta(X)\xi$$

where $P_i = (i = 1, 2)$ are projection maps on the distributions D^\perp and D_θ . Now operating ϕ on both sides of equation (3.1), we have

$$(3.2) \quad \phi X = NP_1 X + TP_2 X + NP_2 X$$

it is easy to see that $TX = TP_2 X$, $NX = NP_1 X + NP_2 X$, $\phi P_1 X = NP_1 X$, $TP_1 X = 0$ and $TP_2 X \in D_\theta$. Also we put

$$(3.3) \quad \phi V = BV + CV$$

for any $V \in T^\perp M$, where BV is the tangent part of ϕV and CV is the normal part of ϕV . We define three tensor fields $\psi_\perp : TM \rightarrow T^\perp M$, $\omega_\theta : TM \rightarrow TM$ and $\kappa_\theta : TM \rightarrow T^\perp M$ by $\psi X = NP_1 X$, $\omega X = TP_2 X$ and $\kappa X = NP_2 X$ respectively, for any $X \in TM$. Now by using all the above equations and the equations of Gauss and Weigarten for the immersion of M in \bar{M} , we obtain following lemmas which play an important role in working out new results.

Lemma 3.1. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} . Then*

$$(3.4) \quad \nabla_X \omega Y - A_{\psi Y} X - A_{\kappa Y} X = \psi \nabla_X Y + \omega \nabla_X Y - Bh(X, Y)$$

$$(3.5) \quad h(X, \omega Y) + \nabla_X^\perp \psi Y + \nabla_X^\perp \kappa Y = \kappa \nabla_X Y + Ch(X, Y)$$

$$(3.6) \quad \eta(\nabla_X \omega Y) = \eta(A_{\psi Y} X) + \eta(A_{\kappa Y} X)$$

for any $X, Y \in TM$.

Proof. For any $X, Y \in TM$, from structure equation we have

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = 0$$

from (3.1), we have

$$\bar{\nabla}_X \phi(P_1 Y + P_2 Y + \eta(Y)\xi) - \phi \bar{\nabla}_X Y = 0$$

using (3.2), we get

$$\bar{\nabla}_X (NP_1 Y + TP_2 Y + NP_2 Y) - \phi \bar{\nabla}_X Y = 0$$

putting the values of tensor fields

$$\bar{\nabla}_X \psi Y + \bar{\nabla}_X \omega Y + \bar{\nabla}_X \kappa Y = 0$$

using Gauss and Weigarten formulae and (3.1), we get

$$\begin{aligned} & -A_{\psi Y} X + \nabla_X^\perp \psi Y + \nabla_X \omega Y + h(X, \omega Y) - A_{\kappa Y} X + \nabla_X^\perp \kappa Y \\ & - \psi \nabla_X Y - \omega \nabla_X Y + \kappa \nabla_X Y - Bh(X, Y) - Ch(X, Y) = 0 \end{aligned}$$

on equating tangential and normal parts, we obtain (3.4) and (3.5).

Lemma 3.2. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} . Then*

- (i) $A_{\phi Z} W = A_{\phi W} Z$ for all $W, Z \in D^\perp$,
- (ii) $[Z, \xi] \in D_\perp$ and $[X, \xi] \in D_\theta$ for all $Z \in D_\perp$ and $X \in D_\theta$,
- (iii) $g([U, V], \xi) = 0$ for all $U, V \in D^\perp \oplus D_\theta$.

Proof. The proof is straightforward and may be obtained by using structure equation with equations (2.4) and (2.5).

Theorem 3.1. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} , then the anti-invariant distribution D_\perp is integrable.*

Proof. For any $Z, W \in D_\perp$ and $Z \in D_\theta$ by using equation (3.1)

$$g([Z, W], TP_2 X) = -g(\phi[Z, W], P_2 X)$$

using Structure equation and (2.4), we get

$$g([Z, W], TP_2 X) = g(A_{\phi Z} W - A_{\phi W} Z, P_2 X)$$

the integrability of distribution follows from Lemma (3.2). \square

Theorem 3.2. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} , then the slant distribution D_θ is integrable*

$$h(X, TY) - h(X, TY) + \nabla_X^\perp NY - \nabla_Y^\perp NX$$

lies in ND_θ for each $X, Y \in D_\theta$.

Proof. For any $Z \in D_\perp$, making use of equations, we obtain

$$g(N[X, Y], NZ) = g(h(X, TY) - h(Y, TX) + \nabla_X^\perp NY - \nabla_Y^\perp NX, NZ)$$

The result follows on using the fact that ND^\perp and ND_θ are mutually perpendicular. \square

Theorem 3.3. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} , then*

- (i) The leaves of the distribution D_\perp are totally geodesic in M if and only if $g(h(D_\perp, D_\perp), ND_\theta) = 0$.

- (ii) The leaves of the distribution D_θ are totally geodesic in M if and only if $g(h(D_\perp, D_\theta), ND_\theta) = 0$.

Proof. (i) By assumption $g(\nabla_Z W, X) = 0$ and $g(\nabla_Z W, \xi) = 0$ for each $Z, W \in D_\perp$ and $X \in D_\theta$, therefore

$$g(\bar{\nabla}_Z \phi W, \phi X) = 0$$

on using Gauss formula

$$g(h(X, \phi Y), NZ) = 0$$

Result follows from above equation.

- (ii) Again by assumption $g(\nabla_X Y, Z) = 0$ and $g(\nabla_X Y, \xi) = 0$ for each $X, Y \in D_\theta$ and $Z \in D_\perp$, therefore using (2.4), (2.5) and structure equation, we get

$$g(h(X, Z), NW) = 0$$

and we complete the theorem. \square

Theorem 3.4. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} , then M is hemi-slant product if and only if*

$$(3.7) \quad \nabla_Z W \in D_\perp$$

for any $Z, W \in D_\perp$.

Proof. Suppose M is a hemi-slant product locally represented by $M_1 \times M_2$. Then M_1 and M_2 are totally geodesic in M then

$$(3.8) \quad \nabla_Z W = \nabla^1_Z W \in D_\perp$$

for any $Z, W \in D_\perp$

$$(3.9) \quad \nabla_X Y = \nabla^2_X Y \in D_\theta$$

for any $X, Y \in D_\theta$

where ∇^1 and ∇^2 are the Riemannian connections on M_1 and M_2 respectively. Again using (3.1)

$$(3.10) \quad g(\nabla_X Z, Y) = -g(Z, \nabla_X Y) = 0$$

for any $X, Y \in D_\theta$ and $Z \in D_\perp$. Thus from (3.8) and (3.10) it follows that (3.7) holds.

Conversely By using the fact that M_1 and M_2 are totally geodesic we get the result. \square

Theorem 3.5. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} . Then M is hemi-slant product if and only if its second fundamental form satisfies*

$$(3.11) \quad Bh(X, Z) = 0$$

$$(3.12) \quad h(X, \phi Y) = Ch(X, Y),$$

for any $X \in TM$ and $Y \in D_\perp$.

Proof. From (3.4) and (3.5) it follows that

$$(3.13) \quad \nabla_X \phi Z = \psi(\nabla_X Z) + Bh(X, Z)$$

and

$$(3.14) \quad h(X, \phi Z) = Ch(X, Z) + \kappa(\nabla_X Z),$$

for any $X \in TM$ and $Z \in D_\perp$. Thus our assertion follows from (3.13) and (3.14) by means of Theorem (3.4).

Now, using the formulas of Gauss and Weigarten, we obtain

$$(3.15) \quad g(A_{\phi X} Z, Y) = -g(Bh(Z, Y), X),$$

for any $Z \in D_\perp$, $Y \in TM$ and $X \in D_\theta$. □

Then, by using above equation and Theorem (3.5), we obtain the following corollary

Corollary 3.1. *Let M be a hemi-slant submanifold of a cosymplectic manifold \bar{M} . Then the following assertions are equivalent to each other:*

- (i) M is hemi-slant product;
- (ii) the fundamental tensors of Weingarten satisfy $A_{\phi X} Z = 0$, for any $X \in D_\theta$ and $Z \in D_\perp$;
- (iii) the second fundamental form of M satisfies $h(Y, \phi Z) = \phi h(Y, Z)$, for any $Z \in D_\perp$ and $Y \in TM$.

To close this section, we recall that bisectional curvature of a cosymplectic manifold \bar{M} is defined by

$$(3.16) \quad S(X, Y) = \bar{R}(X, \phi X, \phi Y, Y),$$

where X and Y are unit vector fields.

4 Some Inequalities for hemi-slant products

In this section, we obtain an equality for the squared norm of the second fundamental form in terms of constant ϕ -sectional curvature for hemi-slant products in cosymplectic manifolds \bar{M} . Also, we have proved an inequality for hemi-slant warped product submanifolds in a cosymplectic manifold \bar{M} and considered the equality case.

Theorem 4.1. *Let M be a hemi-slant product of a cosymplectic manifold \bar{M} . Then:*

$$(4.1) \quad \frac{1}{2}S(Y, Z) = \|h(Y, Z)\|^2 - 1$$

for any unit vector fields $Y \in D_\perp$, $Z \in D_\theta$.

Proof. By using (2.4), (2.7), Corollary 3.1 and structure equation, we get

$$\bar{R}(Y, \phi Y, Z, \phi Z) = g((\nabla_Y^h h)(\phi Y, Z) - g((\nabla_{\phi Y}^h h)(Y, Z), \phi Z)$$

for $Y \in D_\perp$ and $Z \in D_\theta$. From (2.8), we have

$$\begin{aligned} \bar{R}(Y, \phi Y, Z, \phi Z) &= g(\bar{\nabla}_Y h(\phi Y, Z) - h(\nabla_Y \phi Y, Z) - h(\phi Y, \nabla_Y Z), \phi Z) \\ &\quad - g(\bar{\nabla}_{\phi Y} h(Y, Z) - h(\nabla_{\phi Y} Y, Z) - h(Y, \nabla_{\phi Y} Z), \phi Z) \end{aligned}$$

as g is Riemannian metric, we arrive at

$$\begin{aligned} \bar{R}(Y, \phi Y, Z, \phi Z) &= 2 - g(h(\phi Y, Z), \phi \nabla_Y Z) - g(h(\phi Y, Z), \phi h(Y, Z)) \\ &\quad + g(h(Y, Z), \phi \nabla_{\phi Y} Z) + g(h(Y, Z), \phi h(\phi Y, Z)). \end{aligned}$$

Now, using (3.12) and (3.16), we get

$$\frac{1}{2}S(Y, Z) = \|h(Y, Z)\|^2 - 1$$

we complete the proof. \square

Now, we will prove a condition for existence of hemi-slant products in cosymplectic manifolds in terms of ϕ -sectional curvature c .

Theorem 4.2. *There exist no proper hemi-slant products in a cosymplectic manifold $\bar{M}(c)$ with $c \geq 2$.*

Proof. From (2.3) and (4.1), we get

$$(4.2) \quad \|h(X, Z)\|^2 = \frac{c}{2} \|X\|^2 \|Z\|^2$$

for any unit vector fields $X \in D_\theta$ and $Z \in D_\perp$. We get $c \geq 2$ and this completes the proof. \square

Let $\{\xi = e_0, e_1, \dots, e_p, E_1, E_2, \dots, E_n, \phi e_1, \phi e_2, \dots, \phi e_p, \phi E_1, \phi E_2, \dots, \phi E_n\}$ be an orthonormal basis of \bar{M} such that, $\{\xi = e_0, e_1, \dots, e_p, E_1, E_2, \dots, E_n\}$ are tangent to M . Such that $\{\xi = e_0, e_1, \dots, e_p\}$ form an orthonormal frame of D_\perp and $\{E_1, E_2, \dots, E_n\}$ form an orthonormal frame of D_θ (where n is even). We can take $\{\phi e_1, \phi e_2, \dots, \phi e_p, \phi E_1, \phi E_2, \dots, \phi E_n\}$ as orthonormal frame of $T^\perp \bar{M}$.

Theorem 4.3. *Let \bar{M} be a proper hemi-slant products in a cosymplectic manifold $\bar{M}(c)$, with $c \geq 2$. Then*

$$(4.3) \quad \|h\|^2 \geq np(c - 2).$$

Proof. By adopting above frame, we get

$$\|h\|^2 = \sum_{i,j=1}^p \|h(e_i, e_j)\|^2 + \sum_{\alpha,\beta=1}^n \|h(E_\alpha, E_\beta)\|^2 + 2 \sum_{i,\alpha} \|h(e_i, E_\alpha)\|^2$$

$$\begin{aligned}\|h\|^2 &\geq 2 \sum_{i,\alpha} \|h(e_i, E_\alpha)\|^2 \\ \|h\|^2 &\geq np(c-2)\end{aligned}$$

and so, we have (4.3). \square

We prove following lemma to get the inequality for warped product submanifolds.

Lemma 4.1. *Let $M_\theta \times_f M_\perp$, such that $\xi \in TM_\theta$ be a hemi-slant product submanifold of a cosymplectic manifold \bar{M} , then we have*

$$(4.4) \quad g(h(X, Z), FZ) = X(\ln f)g(Z, FZ)$$

$$(4.5) \quad g(h(X, Y), FZ) = 0$$

for $X, Y \in D_\theta$ and $Z, W \in D_\perp$.

Proof. From Gauss formula $g(h(X, Z), FZ) = g(\bar{\nabla}_Z X, FZ) = g(\nabla_Z X, FZ) = X(\ln f)g(Z, FZ)$.

Similarly again by Gauss formula

$$g(h(X, Y), \phi Z) = 0.$$

This proves the Lemma completely. \square

Now, using the above theorem we have the following main result. We are going to obtain an inequality for the squared norm of the second fundamental form in terms of the warping function for hemi-slant warped product submanifold M of a cosymplectic manifold \bar{M} .

Theorem 4.4. *Let $M = M_\theta \times_f M_\perp$ be a hemi-slant warped product submanifold of cosymplectic manifold \bar{M} . Then, the norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq 2c\|\nabla \ln f\|^2,$$

where $\nabla(\ln f)$ is the gradient of $\ln f$ and c is the dimension of M_\perp . If the equality sign holds then M_θ and M_\perp are totally geodesic submanifolds of M .

Proof. By adopting the frame

$$\|h\|^2 = \|h(D_\theta, D_\theta)\|^2 + 2\|h(D_\theta, D_\perp)\|^2 + \|h(D_\perp, D_\perp)\|^2,$$

using (4.4) and (4.5), we get

$$\|h\|^2 = \|h(D_\theta, D_\theta)\|^2 + 2 \sum_{\alpha=1, i=1}^{n,p} \|h(E_\alpha, e_i)\|^2 + \|h(D_\perp, D_\perp)\|^2, \quad (4.6)$$

The first part of theorem follows from above inequality. If the equality sign holds, then from equation (4.6), we get $h(D_\theta, D_\theta) = 0, h(D_\perp, D_\perp) = 0$. Since M_θ and M_\perp are totally geodesic submanifolds of M . This complete the proof. \square

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